Retake Numerical Mathematics 2
April, 2020

Duration: 1 hour per test.

This is an open book exam.
In front of the questions one finds the points. The sum of the points plus 1 gives the end mark for this test in case of a first attempt. In case of a repair we take the minimum of the mark obtained here and a 6.

Test 1

1. [2] Consider the problem $x^3 = d$ for both x and d real where d is given and can be any number in $[-1, 1]$. Determine the absolute condition number of this problem.

2. Consider the linear problem

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1+10^{-20} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}.$$

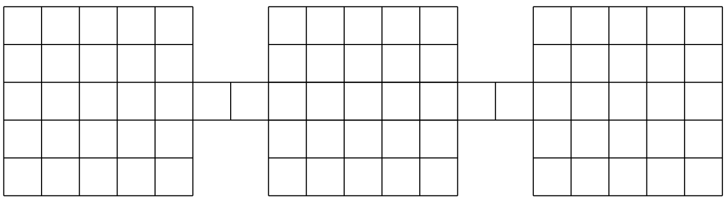
   We want to solve this on a computer which uses a unit round-off of 1e-16.

   (a) [2] Give the solution if we use Gaussian elimination without pivoting.

   (b) [1] Give the solution if we use Gaussian elimination with partial pivoting.

   (c) [1] Which of the two approaches gives the correct approximate solution and why?

3. [3] Consider the graph of a symmetric matrix depicted below. Suppose we cut the problem in three parts by taking the unknowns in the two necks of the problem as separators. Moreover, suppose we order the unknowns in each of the three blocks in lexicographical order. Make a picture of the structure of the resulting linear system and indicate where the unknowns go.



Test 2 an 3 can be found on the other side

Test 2

Consider the matrix

$$\begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & \sqrt{3} \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

1. [2] Give the absolute condition number in 2-norm of the associated eigenvalue problem.
2. [3] Locate the eigenvalues of the matrix using the Gerschgorin circle theorems.
3. [4] Make a QR factorization of the matrix using Householder matrices. Give R explicitly and indicate how one can apply Q by storing only the vectors defining the Housholder matrices. Hint: Note that from linearity we have that $A[\alpha\ 0]^T = \alpha A[1\ 0]^T$.

Test 3

1. Consider the function $f(x)$ on [-1,1] given by

$$f(x) = \begin{cases} 1+x & \text{for } x \in [-1,0], \\ 1-x & \text{for } x \in [0,1]. \end{cases}$$

   (a) [1.5] Let $C_n(x) = \sum_{k=0}^{n} a_k T_k(x)$ be the Chebyshev expansion of $f(x)$. Show that $a_k = 0$ for k odd.

   (b) [3] Compute a_0 and a_2.

   (c) [0.5] Why will $C_n(x)$ converge pointwise to $f(x)$ on the whole interval [-1,1] for n tending to infinity?

2. Consider for arbitrary $f(x)$ the integral $\int_0^\infty \exp(-x) f(x) dx$.

   (a) [3] Derive that the first Gauss rule for this integral is simply $f(1)$.

   (b) [1] Determine the degree of exactness of the rule in the previous part.

Test 1

1 $x^3 = d$ with $-1 \leq d \leq 1$.

$x = d^{1/3}$

$\frac{dx}{dd} = \frac{1}{3} d^{-2/3} \longrightarrow |\Delta x| \leq \max_{|d| \leq 1} \frac{1}{3} d^{-2/3} |\Delta d| = \infty$

So there is no finite abs. condition number. → the problem is unstable.

Without pivoting we get

2 a $\begin{bmatrix} 1 & 10^{-10} & 0 \\ 10^{-100} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 10^{-10} & 0 \\ 0 & -10^{-20} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1-10^{-10} \\ 1 \end{bmatrix}$ *

$$\begin{bmatrix} 1 & 10^{-10} & 0 \\ 0 & -10^{-20} & 1 \\ 0 & 0 & +10^{20} \end{bmatrix} = \begin{bmatrix} 1 \\ 1-10^{-10} \\ 1+10^{20}(1-10^{-10}) \end{bmatrix} \simeq \begin{bmatrix} 1 \\ 1-10^{-10} \\ 10^{20}(1-10^{-10}) \end{bmatrix}$$

$x_3 = 1 - 10^{-10}$

$x_2 = -10^{20}\left[(1-10^{-10}) - x_3\right] = -10^{20}\left[(1-10^{-10}) - (1-10^{-10})\right] = 0$

$x_1 = 1 - 10^{-10} \cdot 0 = 1$

b We start from * and interchange row 2 and 3

$$\left[\begin{array}{ccc|c} 1 & 10^{-10} & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -10^{-20} & 1 & 1-10^{-10} \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 10^{-10} & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1-10^{-10}+\cancel{10^{-20}} \end{array}\right]$$

vanishes through rounding

$$x_3 = 1-10^{-10}$$

$$x_2 = 1$$

$$x_1 = 1 - 10^{-10} x_2 = 1-10^{-10}$$

c The correct answer is given by the one above!
One could check that by computing the residual

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 10^{-10} & 0 \\ 10^{-10} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

for result a we have

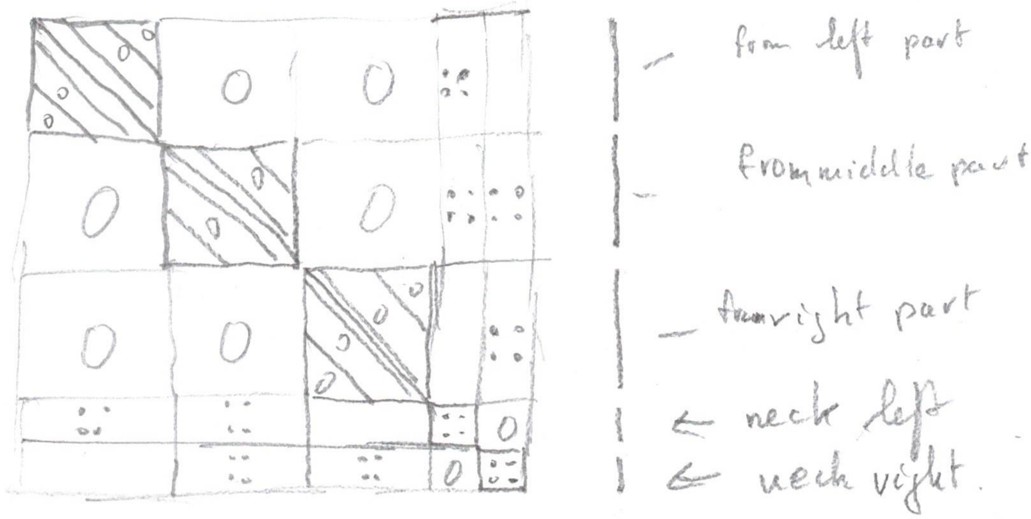
$$\begin{bmatrix} 1-1 \\ 1-10^{-10}-(1-10^{-10}) \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

And for b

$$\begin{bmatrix} 1-(1-10^{-10})-10^{-10} \\ 1-10^{-10}(1-10^{-10})-(1-10^{-10}) \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10^{-20} \\ 0 \end{bmatrix}$$

If no partial pivoting is employed then the original row 3 will be overwhelmed by the second row in *
It's information is lost in the round-off.

3


from left part
frommiddle part
fromright part
neck left
neck right.

Test 2

Consider the matrix

$$\begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & \sqrt{3} \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

a ~~Give~~ the absolute condition number in 2-norm of the associated eigenvalue problem.

Answer

The condition number can be found from the Bauer-Fike theorem saying

$$\min_{\lambda_i \in \sigma(A)} |\lambda - \lambda_i| \le K(P)\|E\|$$

where $P^{-1}AP = D$, for any norm.

Here we have a symmetric matrix, hence P is orthogonal

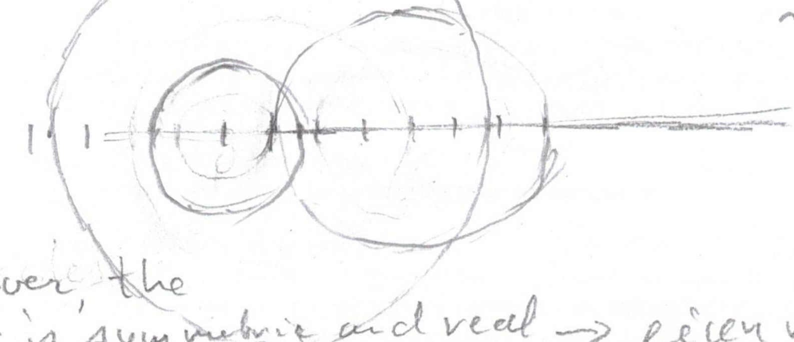
So $\|P\|_2 = \|P^{-1}\|_2 = 1 \rightarrow K_2(P) = 1$

b Locate the eigenvalues of the matrix using the Gershgorin circle theorems.

Answer:

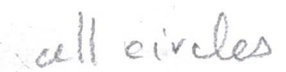
Three circles: $c_1 = 4, r_1 = 3, c_2 = 1, r_2 = 3+\sqrt{3}, c_3 = 0, r_3 = \sqrt{3}$

~ 4.7



Moreover, the matrix is symmetric and real $\rightarrow$ eigen v. are real.

Also the matrix is irreducible since we can get from each unknown to any other unknown.



So since not all circles go through $-2-\sqrt{3}$ and 7 we have that the eigenv. are on $(-2-\sqrt{3}, 7)$

c Make a QR factorization of the matrix using Householder matrices. Give R explicitly and indicate how the Q can be applied knowing the vectors defining the Householder matrices.

Hint: note that

Answer

For the first column we need to mirror

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} \text{ to } \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

The normal of the mirror plane is given by

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \rightarrow v_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\text{So } H_1 = \begin{bmatrix} I - 2v_1v_1^T & 0 \\ 0 & 1 \end{bmatrix}$$

$$(I - 2v_1v_1^T)\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ is in the mirror plane}\right)$$

$$(I - 2v_1v_1^T)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{2}{10}\begin{bmatrix} 1 \\ -3 \end{bmatrix}(-3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\text{So } H_1 \begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & \sqrt{3} \\ 0 & \sqrt{3} & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 3/5 \cdot \sqrt{3} \\ 0 & 1 & -4/5 \cdot \sqrt{3} \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

Next we consider $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ which should be mirrored to $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

hence $\hat{V}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ $\quad v_2 = \frac{\hat{V}_2}{\|\hat{V}_2\|}$

$$H_2 = \begin{bmatrix} 1 & 0 \\ 0 & I - 2v_2v_2^T \end{bmatrix}$$

$$[I - 2v_2v_2^T]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}\frac{1}{2} = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}$$

$$H_2 \begin{bmatrix} 5 & 3 & \frac{3}{5}\sqrt{3} \\ 0 & 1 & -\frac{4}{5}\sqrt{3} \\ 0 & \sqrt{3} & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & \frac{3}{5}\sqrt{3} \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{12}{5} \end{bmatrix}$$

Apparently $Q^T = H_2 H_1 \rightarrow Q = H_1 H_2$

storing v_1 and v_2 is enough to apply H_2 and H_1 successively as above

C $C_n(x)$ converges pointwi[se]
because $f(x)$ is
continuous.

Test 3

Consider the function $f(x)$ which is $\begin{cases} 1+x & \text{for } x \in [-1,0] \\ 1-x & \text{for } x \in [0,1] \end{cases}$

a ~~Show~~ that the $a_k = 0$ for k odd
Note that $T_{2i+1}(\cos\theta) = \cos(2i+1)\theta$
ook uit boek. If $x = \cos\theta$
then $-x = \cos(\pi - \theta)$

$T(x) = T(\cos\theta) = \cos(2i+1)$
$T(-x) = T(\cos(\pi-\theta)) = \cos(2i+1)(\pi-\theta)$
$= \cos((2i+1)\pi - (2i+1)\theta)$
$= \cos(\pi - (\ \ )\theta)$
$= -\cos$
$= -T$

The function is even and also the interval is sym. w.r.t. 0. Moreover, T_{2i} are even and T_{2i+1} are odd. $a_{2i+1} = \dfrac{(f(x), T_{2i+1})}{\|T_{2i+1}\|^2}$

$(f, T_{2i+1}) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f\, T\, dx = 0$
even ev odd

b

So we have
$C_2(x) = a_0 T_0(x) + a_2 T_2(x)$
Min $\|f(x) - C_2(x)\|_2^2$ gives ...
a_0, a_2

$$a_0 = \frac{(f(x), T_0(x))}{\|T_0(x)\|^2}, \quad a_2 = \frac{(f(x), T_2(x))}{\|T_2(x)\|^2}$$

$$(f(x), T_0(x)) = \int_{-1}^{1} \underset{\substack{\uparrow \\ \text{even}}}{\frac{1}{\sqrt{1-x^2}}}\ \underset{\text{even}}{f(x)}, \underset{\text{even}}{T_0(x)}\, dx = 2\int_0^1 \frac{1}{\sqrt{1-x^2}}(1-x)T_0(x)\, dx$$

ow $T_k(\cos y) = \cos(ky)$
sub. $x = \cos y$

$$\int_{\frac{1}{2}\pi}^{0} \frac{1}{\sin y}(1-\cos y)\cdot 1(-\sin y)\, dy = \int_0^{\frac{1}{2}\pi} 1 - \cos y\, dy = \tfrac{1}{2}\pi - \sin y\Big|_0^{\frac{1}{2}\pi}$$
$$= \tfrac{1}{2}\pi - 1$$

idem
$$\|T_0(x)\|^2 = 2\int_0^{\frac{1}{2}\pi} dy = \pi$$

$$\to a_0 = \frac{\pi - 2}{\pi}$$

idem
$$(f(x), T_2(x)) = 2\int_0^{\frac{1}{2}\pi}(1-\cos y)\cos 2y\, dy = 2\int_0^{\frac{1}{2}\pi}\cos y \cos 2y\, dy =$$
$$= \int_0^{\frac{1}{2}\pi} \cos 3y + \cos y\, dy = \tfrac{1}{3}\sin 3y + \sin y\Big|$$

$\cos\alpha+\beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta) = \ldots\ldots + \ldots\ldots$

per ongeluk algemeen uitgerekend.

General expression

$$\int_0^{\frac12\pi}\cos 2ky\,dy=0$$

$$C_{2k}=\frac{\int_0^{\frac12\pi}(1-\cos y)\cos 2ky\,dy}{\int_0^{\frac12\pi}(\cos 2ky)^2\,dy}=\frac{\frac12\int_0^{\frac12\pi}\cos(2k+1)y+\cos(2k-1)y\,dy}{\frac12\int_0^{\frac12\pi}\cos 4ky+1\,dy}$$

$$=\frac{\frac{1}{2k+1}\sin(2k+1)y+\frac{1}{2k-1}\sin(2k-1)y\Big|_0^{\frac12\pi}}{\frac12\pi}=\frac{\frac{1}{2k+1}\sin(k+\frac12)\pi+\frac{1}{2k-1}\sin(k\ldots}{\frac12\pi}$$

$$=\frac{\frac{1}{2k+1}(-1)^k+\frac{1}{2k-1}(-1)^{k+1}}{\frac12\pi}=\frac{\left(\frac{1}{2k+1}-\frac{1}{2k-1}\right)(-1)^k}{\frac12\pi}=$$

$$=\frac{\frac{2}{4k^2-1}(-1)^k}{\frac12\pi}=\left(\frac{1}{\pi(k^2-\frac14)}\right)\cdot(-1)^k$$

$$C(x)=1-\frac{2}{\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{(k^2-\frac14)}(-1)^k T_{2k}(x)$$

$$T_{2k}(0)=T_{2k}(\cos\tfrac12\pi)=\cos 2k\tfrac12\pi=\cos k\pi=(-1)^k$$

$$C(0)=1-\frac{2}{\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2-\frac14}$$

sommeren kun je knap in woorden

$$\frac{1}{k^2-\frac14}=\frac{1}{(k+\frac12)(k+\frac12)}=\frac{1}{k+\frac12}+\frac{1}{k+\frac12}$$

$$\rightarrow\sum_{k=1}^{\infty}\frac{1}{k^2-\frac14}=\frac{1}{k-\frac12}=2$$

$$\rightarrow C(0)=1$$

$C(1)$? $T_{2k}(1)=T_{2k}(\cos 0)=\cos 2k.0=\phi$ (Also $T_{2k}(-1)=1$

$$C(1)=1-\frac{2}{\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2-\frac14}=1-\frac{2}{\pi}-\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2-\frac14}-\frac{1}{(2m)^2-\frac14}$$

dit weet ik niet.

$$(2m-1)^2-\frac14$$

2 a

One needs an accurate numerical integration rule for ~~the~~ integrals of the shape $\int_0^\infty e^{-x} f(x)\,dx$.

~~[illegible]~~ Determine the first Gauss rule

In general we have integration is the exact integration of a polynomial interpolation of $f(x)$

So $f(x) \approx \sum_{i=0}^{n} f(x_i)\, l_i^{(n)}(x)$

where $l_i^{(n)}(x) = \prod_{j=1, j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$

For the Gauss method the interpolation points are the zeros of the associated orthogonal polynomial.

In this case ~~xe~~ the first polynomial with ~~the~~ a zero is $x - \alpha$ where α should be chosen such that $(x-\alpha, 1) = 0$ in the associated inner product.

So $\int_0^\infty e^{-x}(x-\alpha)\,dx = 0 \rightarrow \alpha \underbrace{\int_0^\infty e^{-x}\,dx}_{} = \int_0^\infty e^{-x} x\,dx = \underbrace{-e^{-x} x \Big|_0^\infty}_{0} + \underbrace{\int_0^\infty e^{-x}\,dx}_{\text{same}}$

$\rightarrow \alpha = 1$

α is also the zero so $x_0 = 1$

and $l_0^{(0)}(x) = 1$

So the first integration rule will be

$$\int_0^\infty e^{-x} f(1)\, 1\, dx = f(1) \int_0^\infty e^{-x}\,dx = f(1)$$

$$\int_0^\infty f(x) e^{-x}\,dx \approx f(1)$$

2b Degree of exactness.

To which degree ~~it is~~ polynomials are integrated exactly by the rule. Use that (num.) integ. is linear

degree	polyn	rule	Integral
0	1	1	$\int_0^\infty \exp(-x)\,dx = -e^{-x}\big\vert_0^\infty = 1$
1	x	1	$\int_0^\infty \exp(-x)\,x\,dx = \underbrace{-e^{-x}x\big\vert_0^\infty}_{0} + \int_0^\infty \exp(-x)\,dx = 1$
2	x^2	1	$\int_0^\infty \exp(-x)\,x^2\,dx = \underbrace{-e^{-x}x^2\big\vert_0^\infty}_{0} + 2\underbrace{\int_0^\infty x\exp(-x)\,dx}_{1} =$

not equal

so degree exactness is 1

Alternatively: Gauss has degree of exactness $2n+1$ where n is the number of interp. points here $1 \rightarrow$ degree is 1